Superadditive correlation

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The fact that correlation does not imply causation is well known. Correlation between variables at two sites does not imply that the two sites directly interact, because, e.g., correlation between distant sites may be induced by chaining of correlation between a set of intervening, directly interacting sites. Such ''noncausal correlation'' is well understood in statistical physics: an example is long-range order in spin systems, where spins which have only short-range direct interactions, e.g., the Ising model, display correlation at a distance. It is less well recognized that such long-range ''noncausal'' correlations can in fact be stronger than the magnitude of any causal correlation induced by direct interactions. We call this phenomenon superadditive correlation (SAC). We demonstrate this counterintuitive phenomenon by explicit examples in (i) a model spin system and (ii) a model continuous variable system, where both models are such that two variables have multiple intervening pathways of indirect interaction. We apply the technique known as decimation to explain SAC as an additive, constructive interference phenomenon between the multiple pathways of indirect interaction. We also explain the effect using a definition of the collective mode describing the intervening spin variables. Finally, we show that the SAC effect is mirrored in information theory, and is true for mutual information measures in addition to correlation measures. Generic complex systems typically exhibit multiple pathways of indirect interaction, making SAC a potentially widespread phenomenon. This affects, e.g., attempts to deduce interactions by examination of correlations, as well as, e.g., hierarchical approximation methods for multivariate probability distributions, which introduce parameters based on successive orders of correlation. [S1063-651X(99)07105-6]

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I. INTRODUCTION

Consider the system depicted in Fig. 1. The first degree of freedom, here labeled X_0 for convenience, is connected to M other degrees of freedom X_i, $i=1,\dots,M$, and each of these in turn are connected to the $(M+2)$th, final degree of freedom X_{M+1}. There is no direct connection between site 0 and site $M+1$, however there are multiple, i.e., M, indirect routes of interaction linking site 0 and site $M+1$. For notational convenience we will also reference the variable X_0 as X_h (''h'' stands for ''head'') and the variable X_{M+1} as X_t (''t'' stands for ''tail'').

Interactions are associated with the connections via a Hamiltonian,

$$H=\sum_{i,j=0}^{i,j=M+1} J_{ij}X_iX_j\,, \tag{1}$$

where, in the first class of models we consider, the degrees of freedom X_i are two-state spin variables, taking values ± 1, located at positions indexed by i. In the second class of models that we consider, the degrees of freedom X_i are continuous variables of unrestricted magnitude. The coefficients J_{ij} represent the magnitudes (positive or negative) of the direct connections between these degrees of freedom. Hence, J_{ij} is zero if the variable at i is not connected to the variable at j, e.g., $J_{ht}=0$.

The interactions, $J_{ij}X_iX_j$, induce correlations ρ_{ij} between the variables, which may be expressed in terms of the covariance matrix C_{ij},

$$\rho_{ij}=C_{ij}(C_{ii}C_{jj})^{-1/2}, \tag{2}$$

where

$$C_{ij}=\langle X_iX_j\rangle-\langle X_i\rangle\langle X_j\rangle. \tag{3}$$

SAC is present when the correlation between some variables for which $J_{ij}=0$ is greater than any correlation between any variables for which $J_{ij}\neq 0$. In other words, SAC occurs when the maximal correlation in the system is between variables which are not directly connected. Correlation between vari-

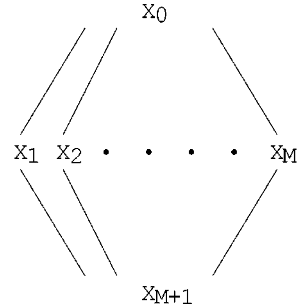


FIG. 1. Architecture of the models. The $M+2$ sites consist of binary spins or continuous variables.

ables that are not directly connected is a well understood effect, called ''long-range order,'' in the physics of spin systems [1]. What is less well known is that correlations involved in long-range order can swamp correlations due to direct connections.

This paper is organized as follows. Section II demonstrates the existence of SAC for a discrete state spin model. This effect is investigated in detail by the method of decimation in Sec. III and by the method of collective coordinates in Sec. IV. The same phenomenon of SAC is then established for a system of continuous degrees of freedom in Sec. V, and again we use decimation and collective variables, in Secs. VI and VII, respectively, for a detailed analysis. For the purpose of clarity, Secs. II–VII are essentially self-contained and may be read independently. Section VIII shows that SAC is also exhibited by mutual information measures and is not a phenomenon restricted to linear correlation measures. Section IX contains discussion and conclusion.

II. SUPERADDITIVE CORRELATION: DISCRETE VARIABLES

Consider $M+2$ binary spins, taking values ± 1. They are labeled $X_0, X_1, X_2, \dots, X_M, X_{M+1}$. We refer to the X_0 and X_{M+1} spins as the ''head'' and ''tail,'' respectively, X_h and X_t. The other spins behave as relays between the head and tail spins. With couplings denoted by J_{ij}, the corresponding Hamiltonian reads

$$H=\sum_{i=1}^{M}(J_{hi}X_h+J_{it}X_t)X_i. \quad (4)$$

The corresponding partition function is

$$Z=\sum_{\{x\}}\exp(-\beta H), \quad (5)$$

where the summation runs over the 2^{M+2} available configurations of spins. Without loss of generality, β, the inverse temperature, can be set equal to 1.

We now restrict, temporarily, the model to the case where all the nonvanishing couplings are equal, $J_{hi}=J_{it}=-J$. Since the thermal averages, $\langle X_i^2\rangle$, of the squared spins are all trivially equal to 1, and the thermal averages of the spins themselves, $\langle X_i\rangle$, are equal to 0, the correlations ρ_{ij} reduce to just the pairwise thermal averages,

$$\rho_{ij}=\langle X_iX_j\rangle=Z^{-1}\sum_{\{x\}}X_iX_j\exp(-H). \quad (6)$$

Of special interest are the cases $M=2$ and $M=3$, which exhibit a transition to superadditive correlation. An elementary calculation shows that, for $M=2$,

$$\frac{\rho_{ht}}{\rho_{h1}}=\frac{e^{2J}-e^{-2J}}{e^{2J}+e^{-2J}}, \quad (7)$$

with the intuitive result that $|\rho_{ht}|<|\rho_{h1}|$, i.e., that the correlation between directly interacting spins is greater than that between nondirectly interacting spins. However, for $M=3$, we find

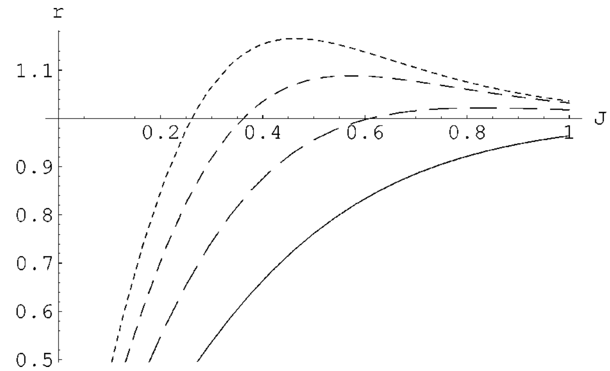


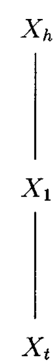
FIG. 2. Binary spin model. Ratio $r=|\rho_{ht}/\rho_{h1}|$ between the noncausal and the causal correlations, as a function of the strength J of the interaction. Full line: case with $M=2$ sites in the intermediate layer. Long dashed line: $M=3$. Dashed line: $M=4$. Dotted line: $M=5$. Notice how r remains <1 for $M=2$, but can be >1 for large values of J when $M\geqslant 3$. The larger M, the easier SAC can occur. However, SAC is maximum for a finite value of J only, which decreases when M increases. Very large values of J damp SAC.

$$\frac{\rho_{ht}}{\rho_{h1}}=\frac{(e^J-e^{-J})(e^{4J}+2e^{2J}+6+2e^{-2J}+e^{-4J})}{(e^J+e^{-J})(e^{2J}+e^{-2J})^2}, \quad (8)$$

with the result that $|\rho_{ht}|>|\rho_{h1}|$ when $|J|$ exceeds a ''critical'' value $J\simeq 0.6$. This effect is illustrated in Fig. 2, where the signature of superadditive correlation is that the ratio $r=|\rho_{ht}/\rho_{h1}|$ is greater than 1. The figure shows the cases $2\leqslant M\leqslant 5$, from which it may be seen that the critical value for SAC is a decreasing function of M. For fixed M, there is an ''optimal'' value of J for which r is maximum. It is seen from Fig. 2 that this optimal value decreases as a function of M, like the critical value. When $J\to\infty$, the ratio r becomes 1. Hence, very large values of J are not efficient for implementing SAC.

III. DECIMATION: DISCRETE VARIABLES

Decimation [2,3] is a technique for simply evaluating the thermal equilibrium properties of spin systems. Equilibrium properties may be determined from the partition function Z. Decimation makes the calculation of Z simple for situations in which there are intervening spins:



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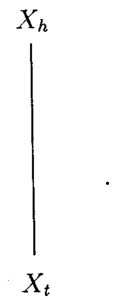
Here X_1 is an intervening spin between X_h and X_t. Decimation works by integrating out the degree of freedom of the intervening spin X_1 and replacing it by an effective direct interaction between X_h and X_t represented by T_{ht},

$$\sum_{X_1} \exp[-(J_{h1}X_hX_1+J_{1t}X_1X_t)]=2K\exp(-T_{ht}X_hX_t). \tag{9}$$

Hence decimation converts the above diagram to



K is a multiplicative constant to the partition function, and T_{ht} represents the effective direct interaction:

$$K^2=\cosh(J_{h1}+J_{1t})\cosh(J_{h1}-J_{1t}), \tag{10}$$

$$T_{ht}=-\frac{1}{2}\ln\left[\frac{\cosh(J_{h1}+J_{1t})}{\cosh(J_{h1}-J_{1t})}\right]. \tag{11}$$

Note that the sign of the effective direct interaction T_{ht} is determined by the relative sign of J_{h1} and J_{1t}. If J_{h1} and J_{1t} have the same sign, as is the case for all intervening spins in the example of the preceding section, then T_{ht} is negative (ferromagnetic). If J_{h1} and J_{1t} have opposite signs, then T_{ht} is positive (antiferromagnetic).

In the following, we will specialize to the case where $J_{h1}=J_{1t}=-J$, in which case the expressions for K and T_{ht} simplify to

$$K^2=\cosh(2J), \tag{12}$$

$$T_{ht}=-\frac{1}{2}\ln[\cosh(2J)]. \tag{13}$$

Decimation, when applied to multiple intervening spins, explains the phenomenon of SAC. Decimating each of the multiple intervening spins in Fig. 3 yields multiple, parallel, direct interactions between the head and tail spins, as represented in Fig. 4. Such parallel direct interactions simply add together, resulting in a single final direct interaction. Hence the effects of even weak interactions between connected spins can add together to make a single strong, effective interaction between nonconnected spins. Depending on the number of intervening spins, the resulting single effective interaction can be stronger than any direct, causal interaction in the system. Although each effective interaction is a nonlinear function of the direct interactions, the process of decimation for multiple intervening spins results in a linear superposition of these nonlinear functions, where each term in the superposition (for our example) has the same sign. SAC may therefore be considered somewhat analogous to additive constructive or destructive interference, a phenomenon familiar in elementary linear wave theory.

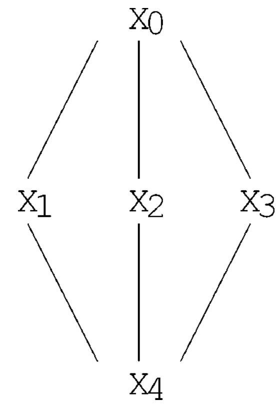


FIG. 3. An example of decimation: the spins X_1, X_2, X_3 will be integrated out.

When $J_{hi}=J_{it}=-J$, the Hamiltonian, Eq. (4), may be written

$$H=-J(X_h+X_t)\left(\sum_{i=1}^{i=M} X_i\right). \tag{14}$$

The ratio, $r=|\rho_{ht}/\rho_{h1}|$, may be easily evaluated by decimation, with the result that

$$r=\frac{[\cosh(2J)]^M-1}{|\sinh(2J)|[\cosh(2J)]^{M-1}}. \tag{15}$$

This was plotted for various values of M and J in Fig. 2.

IV. COLLECTIVE MODE: DISCRETE VARIABLES

Here we want to single out the degrees of freedom X_h, X_t, X_1 in a model where all relay degrees of freedom play a symmetric role. This happens, for instance, when all nonvanishing couplings are equal, namely $J_{hi}=J_{it}=-J$, $i=1,2,\ldots,M$. Then we need only calculate covariances involving X_h, X_t, X_1. Then Eq. (6) reads

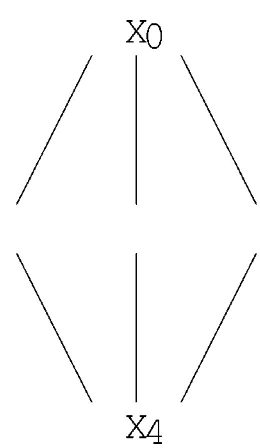


FIG. 4. Additive interactions resulting from decimation.

$$\rho_{ij}=Z^{-1}\sum_{\{3\}} X_iX_j \exp[J(X_h+X_t)X_1] \times\sum_{\{M-1\}} \exp[J(X_h+X_t)(X_2+X_3+\cdots+X_M)], \quad i=h,t,1, \quad j=h,t,1. \tag{16}$$

Here $\Sigma_{\{3\}}$ means the summation over the eight configurations corresponding to X_h, X_t, X_1. Similarly $\Sigma_{\{M-1\}}$ means the summation over the 2^{M-1} configurations corresponding to $X_2, X_3, \ldots, X_M$. Since we are going to examine ratios such as $r=|\rho_{ht}/\rho_{h1}|$, the coefficient Z^{-1}, or any other global coefficient, can be omitted for the present argument.

The decimation represented by the summation $\Sigma_{\{M-1\}}$ can be interpreted as a summation over the integer values taken by the ‘‘collective variable’’ $X_2+X_3+\cdots+X_M$. These range, with increments of 2, between $1-M$ and $M-1$, with occurrence numbers (multiplicities) the traditional binomial coefficients. More explicitly, Eq. (16) reads

$$Z\rho_{ij}=\sum_{\{3\}} X_iX_j \exp[J(X_h+X_t)X_1] \times \sum_{n=0}^{n=M-1} \frac{(M-1)!}{n!(M-1-n)!} \times\exp[J(X_h+X_t)(M-1-2n)]. \tag{17}$$

The ‘‘collective sum’’ gives

$$Z\rho_{ij}=\sum_{\{3\}} X_iX_j \exp[J(X_h+X_t)X_1] \times\exp[(M-1)J(X_h+X_t)] \times\{1+\exp[-2J(X_h+X_t)]\}^{M-1}, \tag{18}$$

and finally

$$2^{1-M}Z\rho_{ij}=\sum_{\{3\}} X_iX_j \exp[J(X_h+X_t)X_1] \times\{\cosh[J(X_h+X_t)]\}^{M-1}. \tag{19}$$

It will be noticed that, since X_h and X_t are binary spins restricted to the values ±1, there is an equivalent form of the same result, namely,

$$\cosh[J(X_h+X_t)]=K\exp(-T_{ht}X_hX_t), \tag{20}$$

where K and T_{ht} are those parameters defined by Eq. (12) and Eq. (13), respectively. The preceding section and the present one thus agree completely.

The strong head-tail correlation brought by the effective interaction $-(M-1)T_{ht}X_hX_t$ is transparent. When $(M-1)T_{ht}\gg1$, it is perfectly safe to disregard here the configurations where $X_hX_t=-1$, since they are penalized by exponentially small Boltzmann factors. Thus ρ_{ht} is *very* close to 1. Only four out of eight configurations are left to calculate $\langle X_hX_1\rangle$. It is trivial to find that ρ_{h1} boils down to tanh(2J). This is a signature for SAC as long as $|\tanh(2J)|$ is not too close to 1, namely as long as J remains moderate. All the qualitative conclusions drawn from Fig. 2 are thus recovered. Of course, the recovery of Eq. (15) from Eq. (19) also implies that all quantitative conclusions are similarly recovered from the collective mode analysis.

V. SUPERADDITIVE CORRELATION: CONTINUOUS VARIABLES

The multivariate Gaussian probability density distribution $P(X)$ describing $M+2$ correlated, continuous variables with covariance matrix C_{ij} and averages μ_i is given by

$$P(X)=(\det C)^{1/2}(2\pi)^{-(M+2)/2} \times\exp\left[-\frac{1}{2}\sum_{ij} C_{ij}^{-1}(X_i-\mu_i)(X_j-\mu_j)\right]. \tag{21}$$

In the above, C_{ij}^{-1} refers to the elements of the inverse matrix of C_{ij}. This may be interpreted as the equilibrium probability density of a system with Hamiltonian, $H=\Sigma_{ij}C_{ij}^{-1}(X_i-\mu_i)(X_j-\mu_j)$, in a heat bath with inverse temperature $\beta=\frac{1}{2}$. Comparing to our previous expression, Eqs. (4) and (5), for the equilibrium probability distribution of a spin system with interaction constants J_{ij}, we see that $J_{ij}=C_{ij}^{-1}$. For continuous variables there are also nonzero self-couplings J_{ii}. Expectations are defined in the usual fashion by integrals over R^{M+2} with a weighting factor given by the probability density $P(X)$ above. The covariance matrix, $C_{ij}=\langle X_iX_j\rangle-\langle X_i\rangle\langle X_j\rangle=\langle X_iX_j\rangle-\mu_i\mu_j$, is related to the correlation matrix ρ_{ij} by

$$\rho_{ij}=C_{ij}(C_{ii}C_{jj})^{-1/2}. \tag{22}$$

The relationship between the interaction matrix J_{ij} and the correlation matrix is therefore straightforward for the continuous case, in contrast to the discrete case. In the following model, we set $\mu_i=0, \forall i$.

A physical model, in terms of points connected by springs, yields the form of the interaction matrix exhibiting SAC for continuous variables. Consider $M+2$ points, each tied to the origin as an anchor position with a spring constant $4k$. Furthermore, consider the zeroth point to be tied to points 1 through M by springs with spring constants equal to 4. Similarly, consider the $(M+1)$th point to be tied to points 1 through M by springs with spring constant equal to 4. The resultant Hamiltonian H is

$$H=2\sum_{i=0}^{i=M+1} kX_i^2+2\sum_{i=1}^{i=M} (X_i-X_0)^2+2\sum_{i=1}^{i=M} (X_i-X_{M+1})^2, \tag{23}$$

which provides a parametrization of J_{ij} in terms of arithmetic constants and the adjustable parameter, k. The correlation matrix is then directly determined in terms of $C_{ij}=J_{ij}^{-1}$, where J_{ij}^{-1} refers to the elements of the inverse matrix of J_{ij}.

To give a numerical example, for $M+2=9$ (seven intermediate variables), then $k=0.35$ yields SAC. The matrix to be inverted is

$$\mathcal{J}=2\begin{bmatrix} 7.35 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & 2.35 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 2.35 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2.35 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 2.35 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 2.35 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 2.35 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2.35 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 7.35 \end{bmatrix}, \qquad (24)$$

with the result $\rho_{h1}=0.56<\rho_{ht}=0.68$. In Sec. VI and Sec. VII we show, using two different methods, that SAC generally occurs for the Hamiltonian, Eq. (23), for (positive) values of M and k related by

$$M^2-kM-k^2-3M-2k>0. \qquad (25)$$

The minimum value of M for which SAC occurs with continuous degrees of freedom is then $M=4$. The associated values of k are $0<k<0.61$. Further analysis is deferred to the following sections.

VI. DECIMATION: CONTINUOUS VARIABLES

Decimation for continuous variables proceeds analagously to that for discrete variables by integrating out the intervening degreees of freedom between head and tail variables. Consider a Hamiltonian describing interactions of a ‘‘head’’ variable X_0, a ‘‘tail’’ variable X_{M+1}, and one intervening variable X_1, denoted here as X_h, X_t, and X_1, respectively. The following Hamiltonian is only a special case of Eq. (23),

$$H=2k(X_h^2+X_t^2+X_1^2)+2(X_h-X_1)^2+2(X_1-X_t)^2. \qquad (26)$$

This may be expanded as

$$H=2(k+1)(X_h^2+X_t^2)+2(k+2)X_1^2-4X_1(X_h+X_t). \qquad (27)$$

The formula for the integration of a Gaussian,

$$\int_{-\infty}^{\infty}dx\,\exp[-ax^2+bx]=\left(\frac{\pi}{a}\right)^{1/2}\exp\left(\frac{b^2}{4a}\right), \qquad (28)$$

allows one to decimate the intervening variable X_1 resulting in

$$\int_{-\infty}^{\infty}dX_1\,\exp\left(-\frac{1}{2}H\right) =\left(\frac{\pi}{k+2}\right)^{1/2}\exp\left[-(k+1)(X_h^2+X_t^2)+\frac{(X_h+X_t)^2}{k+2}\right], \qquad (29)$$

from which we note, upon expanding the $(X_h+X_t)^2$ term, an effective interaction between head and tail variables proportional to $-4/(k+2)$. The magnitude of this effective interaction increases proportionally to the number of intervening variables that are decimated, i.e., $T_{ht}=-4M/(k+2)$ if M variables are decimated. The linear additivity of the effects of multiple intervening variables is therefore simpler in the continuous case compared to the discrete case of Sec. III.

For M intervening variables, H becomes

$$H=2k\left(X_h^2+X_t^2+\sum_{i=1}^{i=M}X_i^2\right) +2\sum_{i=1}^{i=M}(X_h-X_i)^2+2\sum_{i=1}^{i=M}(X_i-X_t)^2. \qquad (30)$$

The relative magnitude of ‘‘head-tail’’ to ‘‘head-intermediate’’ correlations can be evaluated by decimating only $M-1$ of the intervening variables, i.e., decimating $X_2, X_3, \ldots, X_M$, resulting in an expression involving X_h, X_t, and X_1,

$$\int_{-\infty}^{\infty}dX_2\cdots\int_{-\infty}^{\infty}dX_M\,\exp\left(-\frac{1}{2}H\right) \propto\exp\left[\left(-k-M+\frac{M-1}{k+2}\right)(X_h^2+X_t^2)-(k+2)X_1^2\right] \times\exp\left(2(X_h+X_t)X_1+2\frac{M-1}{k+2}X_hX_t\right), \qquad (31)$$

where $\propto$ means up to an inessential multiplicative constant. Comparing the above expression to Eq. (21), one may easily read off elements of the inverse of the covariance matrix as follows: $C_{hh}^{-1}=2[(k+M)-(M-1)/(k+2)]$, $C_{tt}^{-1}=2[(k+M)-(M-1)/(k+2)]$, $C_{ht}^{-1}=-2(M-1)/(k+2)$, $C_{11}^{-1}=2(k+2)$, $C_{h1}^{-1}=-2$, $C_{t1}^{-1}=-2$. Evaluation of the associated correlations requires inversion of the symmetric 3×3 matrix having the above elements. Of interest is only the ratio $r=\rho_{ht}/\rho_{h1}$, where $\rho_{ht}=C_{ht}/(C_{hh}C_{tt})^{1/2}$ and $\rho_{h1}=C_{h1}/(C_{hh}C_{11})^{1/2}$. Squaring r for simplicity, and evaluating the defining inequality for SAC, $r^2>1$, leads to the following condition after some minor algebra:

$$M^2[k^2+(M+2)k+2] > (k+M)(k+2)[k^2+(M+2)k+M]. \tag{32}$$

It may be verifed for $k>0$ and similtaneously $M>0$ that this inequality is satisfied when

$$M^2-kM-k^2-3M-2k>0. \tag{33}$$

The region in (M,k) space where SAC is exhibited is shown in Fig. 5.

VII. COLLECTIVE MODE: CONTINUOUS VARIABLES

Consider Eq. (23) and the corresponding matrix of couplings. For the sake of definiteness in the following displayed matrices, we set $M=5$ temporarily, the generalization to any value of M being obvious. Also, for convenience, the labeling of rows and columns is slightly modified: the order of degrees of freedom in the following matrices is $X_h \equiv X_0, X_t \equiv X_{M+1}, X_1, X_2, \ldots, X_M$, because we want to exhibit the symmetry between the head and the tail and, moreover, we want to stress the matrix block structure which goes with the existence of the intermediate layer and the absence of direct coupling between head and tail. It will be noticed that the formulas which relate covariance to correlation matrix elements are insensitive to an arbitrary scaling of the degrees of freedom. The same is true for the ratios $r_i = |\rho_{ht}/\rho_{hi}|$, $i=1,2,\ldots,M$, which compare the head-tail correlation with any of the head-relay (or tail-relay) correlations.

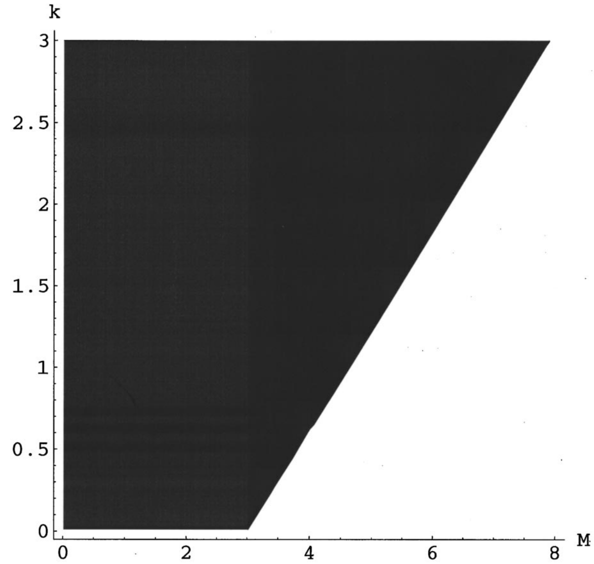


FIG. 5. In the $M>0,k>0$ quadrant, the white area shows where the model with continuous variables exhibits SAC. The dark area, conversely, does not allow SAC.

The matrix to be inverted then reads

$$\mathcal{J}=2\begin{bmatrix} M+k & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & M+k & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2+k & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2+k & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2+k & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2+k & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2+k \end{bmatrix}. \tag{34}$$

While the diagonal matrix elements account for an anchoring of the X_i's to the origin of the configuration space, H, Eq. (23) contains, besides such one-body terms, the two-body interactions

$$V=-2(X_h+X_t)\sum_{i=1}^{i=M} X_i. \tag{35}$$

Such nondiagonal terms represent a dipole-dipole interaction between, on the one hand, ''the head plus the tail,'' and, on the other hand, a collective coordinate $\Sigma_{i=1}^{i=M} X_i$. Out of this collective coordinate, we want to single out, for example, X_1, in order to compare ρ_{ht} with ρ_{h1}. It is then convenient to define

$$\mathcal{X}=(M-1)^{-1/2}\sum_{i=2}^{i=M} X_i, \tag{36}$$

and take advantage of the fact that

$$V=-2(X_h+X_t)[X_1+(M-1)^{1/2}\mathcal{X}]. \tag{37}$$

When expanded with respect to the initial degrees of freedom X_i, the collective coordinate $\mathcal{X}$ can be viewed as a ''symmetric'' vector, $(M-1)^{-1/2}(1,1,1,\ldots,1)$, with equal components in the ''collective'' subspace $\mathcal{C}$ spanned by $X_2, X_3, \ldots, X_M$. The coefficient $(M-1)^{-1/2}$ ensures a proper normalization of $\mathcal{X}$ with a Euclidean metric in $\mathcal{C}$.

For the calculation of ρ_{ht} and ρ_{h1}, nothing prevents us from representing $\mathcal{J}$ on a suitable basis of the subspace $\mathcal{C}$, such as an orthonormal basis including $\mathcal{X}$. For $M=5$, such a basis could, for instance, be made of $\mathcal{X}$ and the other three vectors $(M-1)^{-1/2}(1,1,-1,-1)$, $(M-1)^{-1/2}(1,-1,1,-1)$, and $(M-1)^{-1/2}(1,-1,-1,1)$. Listing such vectors as columns generates the matrix

$$\mathcal{R}=(M-1)^{-1/2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}. \qquad (38)$$

For $M\neq 5$ the structure of $\mathcal{R}$ is the same, with $\mathcal{X}$ in its first column and all other columns showing vectors orthogonal to $\mathcal{X}$. Since X_h, X_t, and X_1 are left intact, the considered change of basis in the $(M+2)$-dimensional space will involve the block matrix,

$$\mathcal{S}=\begin{bmatrix}I_3 & \overline{0}\\ 0^T & \mathcal{R}\end{bmatrix}, \qquad (39)$$

where I_3, $\overline{0}$, and 0^T are, respectively, the identity matrix in three-dimensional space, the $3\times(M-1)$ null matrix, and its transpose. We are now interested in the new representation of $\mathcal{J}$,

$$\mathcal{S}^{-1}\mathcal{J}\mathcal{S}=2\begin{bmatrix}M+k & 0 & -1 & -(M-1)^{1/2} & 0 & 0 & 0\\ 0 & M+k & -1 & -(M-1)^{1/2} & 0 & 0 & 0\\ -1 & -1 & 2+k & 0 & 0 & 0 & 0\\ -(M-1)^{1/2} & -(M-1)^{1/2} & 0 & 2+k & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2+k & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 2+k & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 2+k\end{bmatrix}. \qquad (40)$$

It must be stressed, whether we consider $\mathcal{J}$ or its inverse $\mathcal{J}^{-1}$, that the 3×3 submatrix corresponding to the subspace spanned by X_h, X_t, X_1 is unchanged. This is why we want to calculate ρ_{ht} and ρ_{h1} in the new representation. The same invariance is true for the $(M-1)\times(M-1)$ submatrix corresponding to $X_2, X_3, \ldots, X_M$. Indeed this submatrix was diagonal and is not modified by the transformation described by $\mathcal{R}$. Furthermore, for $\mathcal{J}$, the vanishing couplings of X_1 to the same subspace $\mathcal{C}$ are also left unchanged, obviously.

We now turn to the couplings of X_h (or equivalently X_t) to $\mathcal{C}$. In the initial representation these make a "symmetric" vector $(-1,-1,\ldots,-1)$, proportional to $\mathcal{X}$, and thus orthogonal to all the other basis vectors of the new representation. Hence all the corresponding new matrix elements vanish, except that one which represents the coupling of X_h and $\mathcal{X}$. The corresponding strength becomes $(M-1)^{1/2}$, as predicted by Eq. (37).

Furthermore, the new representation gives an almost diagonal matrix. The correlations to be calculated then demand the inversion of a 4×4 submatrix only,

$$\mathcal{H}_4=2\begin{bmatrix}M+k & 0 & -1 & -(M-1)^{1/2}\\ 0 & M+k & -1 & -(M-1)^{1/2}\\ -1 & -1 & 2+k & 0\\ -(M-1)^{1/2} & -(M-1)^{1/2} & 0 & 2+k\end{bmatrix}. \qquad (41)$$

Straightforward, but slightly cumbersome, this inversion of $\mathcal{H}_4$ provides the desired results,

$$\rho_{ht}=\frac{M}{k^2+(M+2)k+M},$$

$$\rho_{h1}^2=\frac{(k+M)(k+2)}{[k^2+(M+2)k+M][k^2+(M+2)k+2]}. \qquad (42)$$

It is easy to verify that, for $k>0$ and simultaneously $M>0$, the quantity $r^2=(\rho_{ht}/\rho_{h1})^2$ is larger than 1 as soon as

$$M^2-kM-k^2-3M-2k>0. \qquad (43)$$

This criterion comes from the factorization of r^2-1, a rational function of k and M in the model. All factors are positive definite, except that one, $M^2-kM-k^2-3M-2k$. This defines the parameter domain where SAC occurs. The minimal integer value of M for which this condition allows positive values of k is $M=4$. For $M=3$, the roots of the left-hand side of this condition with respect to k are $k=0$ and $k=-5$, while for $M=4$ these are $k=-6.61$ and $k=0.61$. As shown by Fig. 5, where the area $r<1$ is shaded, the positive k root, showing the border of the SAC domain, increases almost linearly as a function of M. All the results of the preceding section are recovered.

VIII. MUTUAL INFORMATION

In this section we show that our conclusions concerning SAC also hold if mutual information, instead of correlation, is used to quantify the relationship between discrete head, tail, and intermediate variables. Mutual information [4] is defined in terms of entropies as follows:

$$M=\Omega(i)+\Omega(j)-\Omega(ij), \qquad (44)$$

where $\Omega(i)$, respectively $\Omega(j)$, are the single site entropies at position i, respectively j, e.g., $\Omega(i)=-\Sigma_{X_i}P(X_i)\ln P(X_i)$. The pairwise entropy is similarly defined as $\Omega(ij)=-\Sigma_{X_i,X_j}P(X_i,X_j)\ln P(X_i,X_j)$.

The single site and pairwise probability distributions may be related to correlations by the Bahadur-Lazarfeld expansion [5] for an arbitrary, discrete state, multivariate probability distribution,

$$P(X_1,X_2,\ldots,X_N) = P^{\text{indep}}(X_1,X_2,\ldots,X_N) \times\left[1+\sum_{i<j}\rho_{ij}Y_iY_j+\sum_{i<j<k}\rho_{ijk}Y_iY_jY_k+\cdots\right]. \tag{45}$$

In the above, P^{indep} is the independent probability distribution defined by

$$P^{\text{indep}}(X_1,X_2,\ldots,X_N)=\prod_i P_i^{(1+X_i)/2}(1-P_i)^{(1-X_i)/2}, \tag{46}$$

where P_i is the probability that $X_i=1$. The variables Y_i are zero mean, unit variance variables related to the X_i by $Y_i=(X_i-\langle X_i\rangle)/(1-\langle X_i\rangle^2)^{1/2}$. The ρ_{ij} are the usual two-point correlation functions, while $\rho_{ijk},\rho_{ijkl}\ldots$ are similarly three-point, four-point, ..., N-point correlation functions.

It is easily verified from Eq. (45) that the two-point marginals $P(X_i,X_j)$ reduce to

$$P(X_i,X_j)=P^{\text{indep}}(X_i,X_j)\left[1+\sum_{i<j}\rho_{ij}Y_iY_j\right], \tag{47}$$

while the one-point marginals, $P(X_i)$, reduce to $P(X_i)=X_iP_i+(1-X_i)/2$. For the situation considered in Secs. II–IV, $P_i=1/2$, hence evaluating the marginal distributions in this case, and substituting into Eq. (44), which defines the mutual information, yields

$$M_{ij}=1+[(1+\rho_{ij})/2]\ln[(1+\rho_{ij})/2] +[(1-\rho_{ij})/2]\ln[(1-\rho_{ij})/2]. \tag{48}$$

Since $(1+\rho_{ij})/2$ is bounded between 0 and 1, we may define an ''entropy'' $\Omega=-q\ln q-(1-q)\ln(1-q)$ in terms of a fictitious probability $q=(1+\rho_{ij})/2$ and rewrite the above as

$$M_{ij}=1-\Omega(q). \tag{49}$$

In Secs. II–IV, the correlation ρ_{ht} was calculated and compared to ρ_{h1}. SAC is defined by $|\rho_{ht}|>|\rho_{h1}|$. Since entropy is a concave function of the ''probability,'' then Eq. (48) verifies that $M_{ht}>M_{h1}$ whenever $|\rho_{ht}|>|\rho_{h1}|$. Hence, the examples of SAC described in Secs. II–IV hold if mutual information, instead of correlation, is used to quantify the relation between variables.

IX. DISCUSSION AND CONCLUSION

It is well known that multiple routes of communication, besides redundant coding of messages, can ensure reliable transmission of information in the presence of noise. Information transmission is achieved by the preservation of correlation, or mutual information, between an information source and an information receiver. This conservation of information should not be confused with reinforcement of amplitudes, because information is amplitude independent. This paper shows that noisy signals transmitted via multiple routes can preserve information using *coherence effects*.

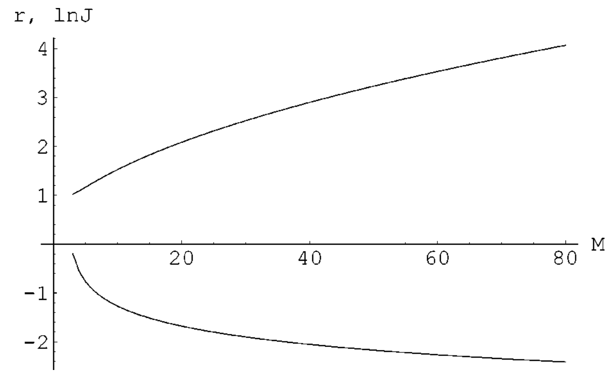


FIG. 6. Best performance r (upper curve) and logarithm of the corresponding coupling strength J (lower curve) as functions of the number M of relays.

As usual, the multiplication of routes has the benefit of robustness, but the defect of an increase of cost. The cost increase, in our case, can be moderated by the use of links with weak, and hence perhaps cheaper, couplings as long as the links remain coherent with one another. While the build up of an amplitude by the addition of coherent signals is not a new phenomenon, the point of this paper is the build up of information via multiple routes involving relays.

What this paper showed in some detail is that the multiple route solution may easily proceed by relays, see Fig. 1 and Fig. 3, rather than by multiple direct connections, see Fig. 4. The method of decimation shows that, all told, relays amount to direct routes, see again Fig. 4. Moreover, this paper studied the occurrence of superadditive correlation at finite temperature. The increased correlation between emitter and the receiver can thus be implemented in the presence of noise. The main result of the paper is that, given a scale for the noise (temperature), the system can be optimized with respect to its cost, namely both its architecture (the number M of relays) and the strength of the couplings J to be implemented. This is clear from Fig. 2, for instance, which illustrates the existence of both a minimum coupling strength for superadditive correlation to occur, and an optimal value of this strength. We show in Fig. 6 the best performance r available and the corresponding optimal J, as functions of M.

A system which shows a great amount of ''fan out–fan in'' architectures is the central nervous system of vertebrates. As shown by [6], synaptic processes are stochastic. There is thus a significant amount of information loss at synapses. It is thus tempting to see whether multiple synaptic contacts between two neurons, or a sensor cell and a neuron, or a neuron and an effector cell, may turn out to restore a correlation which was degraded at individual synapses. Neurophysiological experiments for such tests might be possible. At a larger scale, the very strong ''fan out–fan in'' exhibited by the cerebellum, from mossy fibers to Purkinje cells (as many as 10^5 parallel fibers contacting a Purkinje cell), might also give rise to such reinforcement of correlations, most useful in the coordination of ballistic motions. For early work on the theory of the cerebellum, see [7]. The narrow time windows discovered by [8] in the action of basket cells upon Purkinje cells give a hint that precise time correlations are involved in the task of the cerebellum. A time-dependent reformulation of the present paper is clearly in order.

Finally, the SAC effect raises a warning: while analysis of complex systems often deduces a hierarchy of interactions from a hierarchy of correlations, see, e.g., [9,10], we proved that indirect interactions may generate dominant correlations. The hierarchical approach, therefore, demands some caution. An alternative approach is explored in [11,12].

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